THE

PHYSICAL SOCIETY

OF

LONDON.

PROCEEDINGS.

VOLUME XXX.—PART IV.

JUNE 15, 1918.

Price to Non-Fellows, **4s.** *net, post free* **4/3.**

Annual Subscription, **20/-** *post free, payable in advance*

...nthly from December to August.

LONDON:
FLEETWAY PRESS, LTD.,
1, 2 AND 3, SALISBURY COURT, FLEET STREET,

1918.

THE PHYSICAL SOCIETY OF LONDON.

1918-19.

OFFICERS AND COUNCIL.

President.

PROF. C. H. LEES, D.Sc., F.R.S.

Vice-Presidents:

WHO HAVE FILLED THE OFFICE OF PRESIDENT.

PROF. G. C. FOSTER, D.Sc., LL.D., F.R.S.
PROF. R. B. CLIFTON, M.A., F.R.S.
PROF. A. W. REINOLD, C.B., M.A., F.R.S.
SIR W. DE W. ABNEY, R.E., K.C.B., D.C.L., F.R.S.
PRIN. SIR OLIVER J. LODGE, D.Sc., LL.D., F.R.S.
SIR R. T. GLAZEBROOK, C.B., D.Sc., F.R.S.
PROF. J. PERRY, D.Sc., F.R.S.
C. CHREE, Sc.D., LL.D., F.R.S.
PROF. H. L. CALLENDAR, M.A., LL.D., F.R.S.
PROF. A. SCHUSTER, PH.D., Sc.D., F.R.S.
SIR J. J. THOMSON, O.M., D.Sc., F.R.S.
PROF. C. VERNON BOYS, F.R.S.

Vice-Presidents.

PROF. J. W. NICHOLSON, M.A., D.Sc., F.R.S.
PROF. O. W. RICHARDSON, M.A., D.Sc., F.R.S.
S. W. J. SMITH, M.A., D.Sc., F.R.S.
W. E. SUMPNER, D.Sc.

Secretaries.

PROF. W. ECCLES, D.Sc.,
City and Guilds Technical College, Leonard Street, E.C. 2.

H. S. ALLEN, M.A., D.Sc.
5, *Presburg Road, Malden, Surrey.*

Foreign Secretary.

SIR R. T. GLAZEBROOK, C.B., D.Sc., F.R.S.

Treasurer.

W. R. COOPER, M.A., B.Sc.
82, *Victoria Street, S.W.* 1.

Librarian.

S. W. J. SMITH, M.A., D.Sc., F.R.S.
Imperial College of Science and Technology.

Other Members of Council.

PROF. E. H. BARTON, D.Sc., F.R.S.
C. R. DARLING, F.I.C.
PROF. G. W. O. HOWE, D.Sc.
D. OWEN, D.Sc.
C. C. PATERSON.
C. E. S. PHILLIPS, F.R.S.E.
S. RUSS, M.A., D.Sc.
T. SMITH, B.A.
F. J. W. WHIPPLE, M.A.

XVI. *On the Times of Sudden Commencement of Magnetic Storms.* *By* S. Chapman.

Communicated by Dr. C. Chree, F.R.S.

Received March 22, 1918, and, in revised form, April 13, 1918.

§ 1. As at first communicated, this Paper was written in April 1917, or thereabouts. Some revision has seemed desirable, partly in view of a subsequent change in my ideas concerning the nature of magnetic storms, and partly because a discussion of the same data by Dr. Chree,* with which I was not then acquainted, rendered certain introductory matter superfluous.

In 1910 a Paper by Dr. Bauer† renewed interest in the question as to how far magnetic storms commence simultaneously at different parts of the earth. W. G. Adams and W. Ellis‡ had shown, long before, that the commencement-time is nearly the same, in great magnetic storms, over a large part of the earth; as regards intervals of time amounting to a few minutes, however, the question was still open. Dr. Bauer claimed in his Paper to have demonstrated that the time of propagation "round" the earth was actually of the order of 3 or 4 minutes, and he indicated the supposed direction and speed of propagation in the cases of several particular storms. These conclusions excited some controversy, being contested by various authors.§ In order to assist in settling the point, Dr. Bauer collected, with the aid of the directors of 32 contributing observatories, new data for the commencement-times of 15 magnetic storms.|| This set of material is the amplest and most suitable which has so far been available for the discussion of the subject, and there can be no doubt that its compilation has increased the interest and care taken in the accurate time-scaling of magnetograph registers.

* C. Chree, "Proc." Phys. Soc., XXVI., p. 137, 1914; also *ibid.*, XXIII., p. 49, 1910; and "Nature," LXXXVI., p. 79, 1911.

† L. A. Bauer, "Terr. Mag.," XV., p. 9 (also pp. 107, 221), 1910; "Nature," LXXXV., p. 306, LXXXVI., p. 9, 1911.

‡ W. G. Adams, "Brit. Ass. Rep.," 1880, p. 201, 1881, p. 463; W. Ellis, "Nature," XXIII., p. 33, 1880, "Proc." Roy. Soc., LII., p. 191, 1892; also van Bemmelen, "Amst. Acad. Sci.," Versl., XV., p. 250, 1906, Proc., IX., p. 266, 1906, Proc., 1908; also Birkeland, Norwegian Aur. Pol. Exp., 1902-3, I., p. 63, 1908.

§ Faris, "Terr. Mag.," XV., pp. 93, 213, 1910, and XVI., p. 109, 1911; Chree, *l.c.*; Krogness, "Nature," LXXXV., p. 170, 1910; Walker, *ibid.*, p. 236, 1910; van Dijk, *ibid.*, LXXXVI., p. 44, 1911; Birkeland, *ibid.*, LXXXVI., p. 79, 1911.

|| "Terr. Mag.," XVI., pp. 85, 163, 1911.

Up to this point the discussion seemed to have terminated without bringing about agreement between the opposing sides. The verdict of Dr. Chree and Dr. Krogness was that Dr. Bauer's observational conclusions were not established, so that also the theory which he based upon them could not be maintained.

No discussion of the subsequent collection of data has appeared in "Terrestrial Magnetism," and at the time of my own examination of it I was unaware that the material had already been discussed by Dr. Angenheister* and Dr. Chree.† So far as regards the above-mentioned theory, the three discussions agree in concluding that the new data afford no evidence in its favour, but rather to the contrary. In view, especially, of Dr. Chree's very complete treatment it would be useless to labour this point further; as regards the discussion of instrumental points, and the grouping of the time-data according to latitude and longitude, a general reference to his Paper will suffice. The justification for bringing forward this further discussion must rest upon the novelty of the point of view underlying it, which is different from that adopted by either Dr. Angenheister or Dr. Chree. Although no definite positive results are here derived from the existing data, the method of treatment does, I think, give promise of such when more material is forthcoming.

§ 2. In the various Papers cited only three possibilities seem to have been contemplated regarding the commencement-times. Thus, Dr. Chree‡ remarks: "As regards these 'sudden' changes three things are conceivable: they may be absolutely simultaneous at different stations; there may be a very small difference of time, corresponding to the rate of propagation of electromagnetic waves; or, finally, there may be, as Dr. Bauer concludes, longer intervals, amounting to several minutes, for stations remote from one another. Under existing conditions of registration one cannot decide between the first two possibilities. As between these two and the third, a decision should not be impossible."

While reading Mr. Maunder's Papers§ on the recurrences of magnetic storms at intervals equal to the synodic rotation-

* Angenheister, "Nach. d. K. Ges. d. Wiss. zu Göttingen, Math.-Phys. Kl.," 1913.

† Chree, *l.c.* (1914).

‡ C. Chree, Proc. Phys. Soc., XXIII., p. 49, 1910.

§ E. W. Maunder, "M.N.R.A.S.," Nov. 1904, April, May, 1905, Nov. 1915; "Journ. Brit. Ast. Ass.," XVI., p. 140, 1906; "Astrophysical Journ.," XXI., p. 101, 1905.

period of the sun (27·3 days), a fourth possibility naturally suggested itself. Mr. Maunder's results suggest that storms are occasioned by some solar agent which is transmitted along narrow, well-defined streams, issuing from and rotating with the same period as the sun. The direction of solar rotation is such that the streams, if suitably situated, will overtake the earth on the afternoon or P.M. side.* Further, since the earth's angular diameter, viewed from the sun, is only $18''\cdot 6$, it is easy to calculate that the time necessary for a stream to sweep right across the globe is almost exactly 30 seconds.

The above considerations suffice to explain how, without regard to any particular theory as to the precise mechanism of magnetic storms, the presumption arose that the relative time of commencement of a storm, at different stations, depends mainly on the orientation of the latter, at the time, relative to the sun. The determining factor, it is thus suggested, is neither mainly latitude, nor "absolute" longitude, but principally longitude relative to the sun. Now this relative longitude depends simply on the local time† at the station at the commencement of the storm. The local times of commencement, therefore, form the basis of classification of the data in the present discussion.

§ 3. Again, the *order of magnitude* of the time-differences to be expected, between the commencements at different stations, is suggested by the time (30 seconds) taken for a stream to sweep across the earth. The differences, therefore, are to be reckoned in fractions of a minute, or in seconds, and not, as on the second and third of the hypotheses mentioned by Dr. Chree (§ 2) in small fractions of a second, or in whole

* If we imagine ourselves looking from the sun towards the earth, the latter will be seen to travel along its orbit from right to left. The solar streamers rotate in the same direction, but, moving much more rapidly, overtake the earth. The earth and sun revolve in the same direction, so that the hemisphere visible from the sun (the "day" hemisphere) will be turning from left to right. The local time for all stations to the right of the terrestrial meridian plane containing the sun will be *after* noon, so that the right-hand hemisphere will be termed the P.M hemisphere (*post meridiem*). In this paper it will be assumed, for simplicity, that the axes of earth and sun are perpendicular to the ecliptic, or plane of the earth's orbit. The earth will also be regarded as divided into four quadrants relative to the position of the sun at the time, the local time in the quadrants, which we may respectively term the night A.M., the day A.M., the day P.M., and the night P.M., ranging from midnight to 6 a.m., 6 a.m. to midday, midday to 6 p.m., and from 6 p.m. to midnight.

† Local time measures the longitude, at the rate of 15° per hour, from the "midnight" meridian, which is in the plane containing the sun, but on the dark side of the earth.

minutes. It is true that an electromagnetic change occurring at any station will produce *some* effect, propagated with the speed of light, at all other stations on the earth, but the magnitude of the effect diminishes so rapidly with increasing distance from the source that it soon becomes inappreciable. The initial impulse of a magnetic storm, like the diurnal magnetic variations, is mainly produced by atmospheric currents situated in the comparatively near neighbourhood of each station. The time differences of commencement at different stations are due to the lack of simultaneity of arrival of the solar agent of excitation at those places.

Since this discussion was first completed, having carefully studied the nature of magnetic storms, my then vague ideas have given place to fairly definite conclusions regarding their origin and mechanism.* These do not enable me, however, to conclude any more definitely than at first whether the *precise* range of commencement-time to be expected, in the occurrence of a storm at stations distributed over the earth, is likely to be rather more or rather less than the above 30-second interval. The electric corpuscles which compose the streams do not impinge directly upon the atmosphere, but are deflected by the earth's magnetic field, while still at distances comparable with the earth's diameter. Prof. Störmer † has made calculations as to the paths of single electric corpuscles in the earth's field, and has found that the variety of possible types is very great. His calculations appear to be too ideally simplified, as yet, to give results accurately representative of the actual conditions, the mathematical difficulties even in the simple case of one electron being very formidable. The magnetic storms themselves indicate that the particles are precipitated more upon the P.M. than upon the A.M. hemisphere,‡ and as the former hemisphere is also the one on the side of approach of the streams, it is not unlikely that the initial impulse of a storm occurs first in that hemisphere. The discussion of the data of this Paper seems to favour this view, though the evidence is too slight to warrant a definite conclusion. It may be said, therefore, that at present theory gives but little guide as to the precise results to be expected, and that more

* These conclusions are described in a Paper communicated to the Royal Society on April 10, 1918.

† Störmer, "Arch. d. sc. phys. et nat. Genéve," 1907, and various articles in "Terrestrial Magnetism" and elsewhere.

‡ Cf. the footnote to p. 207; also the Paper on magnetic storms referred to in the last footnote but one.

observational material is required in order to ascertain the actual facts.

§ 4. The data to be considered are confined to the times of commencement as measured from the horizontal force record alone. The sudden perturbation at the beginning of a storm is generally most pronounced, and therefore most accurately measurable, in this element. The principal source of error in the data appears to be the uncertainty as to which is the precise movement on the registers which marks the sudden commencement. If proper precautions are taken, the time of a given movement on the record is measurable with an accuracy of about a minute. An inspection of the data suggested that (with perhaps one exception) the *systematic* error of time determination, at any of the 32 observatories which contributed data, did not exceed one minute. I therefore decided to reject any observation which stood out from the mean time of commencement for the particular storm by two minutes or more, on the ground that the difference probably arose from a choice of the wrong movement for measurement. Some observatories gave more than one commencement-time for one or two of the storms, accompanied by drawings showing the magnetic movements to which the times corresponded. In these cases it was usually possible to choose, with little uncertainty, the precise movement which had been taken by the majority of the other observatories. The total number of rejected values was not large,* while the number retained was 342 ; if all the 32 observatories had given data for each storm, the total number of values would have been 480, but some observatories did not furnish complete data. The table (facing p. 212) shows the rejected values as well as those retained.

The mean time of commencement for each storm was determined from the values given by the several observatories, excluding those which were rejected. The residual differences, "actual value—mean," are given in the table, for all the contributing observatories. Where a query is attached to an observation, this was done by the observatory which supplied it, and these values have in all cases been bracketted. A few other particulars regarding the table are indicated in the notes appended to it. The observatories are arranged in order

* Counting those rejected on ground of magnitude alone, there were 31 negative and 22 positive residuals bracketted.

of longitude, and the storms in order of date. Columns are added giving the systematic difference of time from the mean, and also the mean numerical residual, for each *observatory*; in a row beneath, the mean numerical residual for each *storm* is given. The number of residuals concerned in each mean is indicated by a suffix. The unit in which all these* quantities are expressed is one-tenth of a minute.

For some storms the data are clearly much better than for others; this is probably owing to the greater definiteness of the initial perturbation in the former cases. Magnetic storms show marked differences in this particular. Storm 1 is an example of a "good" storm—only two values are rejected out of 27, and the mean numerical residual of the other 25 is 0·4 minute. The numerical mean residual derived from the 342 values all taken together is about 0·7 or 0·8 minute. This should allow a clear indication to be given of times of the order of two or three minutes, such as were suggested by Dr. Bauer; this, it will be seen, is not shown, the residuals, which are of about one minute in magnitude, largely consisting of accidental error. When it is remembered that, with the ordinary time-scale, 1 mm. on a magnetograph sheet represents about three minutes of time, this degree of accidental error is not surprising.

§ 5. In accordance with the ideas described in §§ 2, 3, the residuals for each storm were divided up into four groups, according to which of the four quadrants mentioned in the footnote to § 2 the corresponding station was situated in at the time of commencement. This was done by considering the longitudes, west of Greenwich, of the four principal meridians with respect to the sun, viz., the midnight, midday, 6 a.m. and 6 p.m. (local time) meridians. Thus, in storm 1, the commencement occurred at 19h 56·7m G.M.T., so that the midnight meridian was this amount (which in arc is 299°·2) to the west of Greenwich. The longitudes of the four meridians (0, 6, 12 and 18 hours local time) were therefore 299°·2, 209°·2, 119°·2, and 29°·2. In the table dividing lines representing these longitudes were drawn for the first storm (and in a similar fashion for the other storms), these lines being respectively light and continuous, broken, thick and continuous, and broken. The stations in the night A.M. quadrant at the

* At the foot of the table, however, are other quantities, described in § 5, which are expressed in units of 1 second.

commencement of any storm are consequently those above the light continuous line and below the consecutive (broken) line. In this way it becomes an easy matter to group the residuals in the table according to their corresponding quadrant or hemisphere.

At the foot of the table rows are to be found which give for each storm the means of the residuals in the four hemispheres, day, night, P.M. and A.M., and also the differences between the results for complementary hemispheres. These values are all given in units of 1 second, not 0·1 minute as in the remainder of the table. The mean difference for either pair of hemispheres in the case of no storm exceeds 60 seconds, and these mean differences for individual storms are clearly still affected by considerable accidental error. Any systematic quantities underlying them must, therefore, be of the order of 20 seconds or less. Taking all the residuals in any one quadrant together, we obtain the following results. The number of residuals, their sum* (in minutes) and their mean (in seconds), are given in order :—

Night P.M.	quadrant	(84) :	Sum	−5·0m.,	Mean	−3·6s.
Day P.M.	,,	(89) :	,,	−3·8m.,	,,	−2·6s.
Night A.M.	,,	(87) :	,,	−1·1m.,	,,	−0·8s.
Day A.M.	,,	(82) :	,,	+7·2m.,	,,	+5·3s.

The quadrants are arranged in the numerically increasing order of their mean residuals. This order is not to be regarded as the true order of arrival of the initial impulse of a magnetic storm at the different quadrants; more data would be needed to establish any such conclusion. There is perhaps enough evidence to favour the opinion that the P.M. hemisphere is the one first affected by the storm, though even this is doubtful. If we group the quadrants into pairs of hemispheres, we obtain the following results :—

P.M. residuals	(173) :	Sum −8·8m. :	Mean −3·2s.	} P.M.–A.M., −5·3s.
A.M. ,,	(169) :	,, +6·1m. :	,, +2·2s.	

Day residuals	(171) :	Sum +3·4m. :	Mean +1·2s.	} Night−day, −3·3s.
Night ,,	(171) :	,, −6·1m. :	,, −2·1s.	

The latter result is certainly more unlikely than the former, but cannot be dismissed as (theoretically) incredible. In order to test how far accidental error might account for these small

* It may be noted that the sum of all the residuals is −2·7m., which arises from an accumulation of small errors, less than 0·1m., in the adopted mean times of commencement of the storms.

differences of time, I made the following two quite arbitrary subdivisions of the data :—

First 16 observatories (172 residuals) : Sum −1·9m. : Mean −0·7s. Second 16 observations (170 residuals) : Sum −0·8m. : Mean −0·3s.	Difference 0·4s.

Upper right-hand diagonal half (177 residuals) : Sum +2·5m. : Mean +0·8s. Lower left-hand diagonal half (165 residuals) : Sum −5·2m. : Mean −1·9s.	Difference 2·7s.

The latter, which is about equal to the difference between the day and night residuals, confirms the unlikelihood of the latter being real. The matter for surprise in all cases, however, is the extreme smallness of the residuals, and this suggests that it should not be very difficult, even with present instrumental means, to determine any systematic time differences between the different quadrants, if these amount to so much as 10 seconds. The extreme range (30 seconds, more or less) would not, of course, show in the mean of whole quadrants or hemispheres ; and the fact that the majority of observatories contributing data are in high latitudes likewise tends to minimise the differences to be expected.

ABSTRACT.

The Paper is a discussion from a new standpoint of the data, collected by Dr. Bauer, for 15 magnetic storms. Maunder's work on the recurrence of magnetic storms at intervals equal to the rotation period of the sun suggests that storms are due to some solar agent transmitted along narrow well-defined streams, issuing from and rotating with the sun. This suggests the view that the relative time of commencement of a storm at different stations depends mainly on the orientation of the latter at the time, relative to the sun, *i.e.* on the local time at the station. This forms the basis of the classification in the Paper.

DISCUSSION.

Dr. Chree said that after hearing the Paper he was uncertain whether Dr. Chapman did or did not believe that the figures *proved* anything. He (Dr. Chree) did not think that they did. To deal with a question of a few seconds' difference at different stations, it would, he thought, be desirable to employ not merely a more open time scale, but magnetographs of greater and desirably uniform sensitiveness. When a magnetic change became recognisable depended on the size of the movement and the sensitiveness of the magnetograph. An apparent difference in time between day and night hemispheres would naturally arise if movements tended to be larger in the one than the other. One objection which he had urged against Dr. Bauer's views also applied to the hypothesis considered by Dr. Chapman, viz., that as soon as a disturbance began at any part of the earth's atmosphere it would naturally be propagated b

Observatory.	West longitude.	Latitude.	Index number of storm, wit						
			1	2	3	4	5	6	7
			1906, July 29. 19h56·7m	1906, Aug. 7. 13h38 1m	1906, Dec. 21. 21h30·6m	1907, Feb. 9. 14h13·0m	1907, July 10. 14h22·7m	1907, Oct. 13. 7h43·2m	1908, Mar. 26. 17h41·4m
	°	°							
Greenwich*	0·0	51·5 N	3	(−37)	−6	10	3	8	(16)‡
Kew*............	0·3	51·5 ,,	3	9	4	0	−7	−2	(16)‡
Stonyhurst	2·5	53·8 ,,	−10	−6	(−25)	(−21)	−2	−5	...
Eskdalemuir....	3·2	55·3 ,,	...	...	...	...	...	...	...
Falmouth*	5·1	50·1 ,,	3	−1	(34)	(20)	13	18	(26)
Pilar	63·9	31·7 S	...	...	−5	(25)	9	−6	11
Porto Rico	65·4	18·1 N	−3	−14	(34)	1	−1	2	0
Cheltenham	76·8	38·7 ,,	−2	9	12	2	0	−2	(36?)
Agincourt	79·4	43·7 ,,	−4	(−20)	4	−5	−2	0	14
Baldwin	95·2	38·8 ,,	4	11	0	0	5	1	8
Sitka	135·3	57·0 ,,	−7	(21)	0	5	−3	(8?)	...
Honolulu	158·1	21·3 ,,	(−24)	−9	(−23)	−10	−18	−11	−12
Samoa	171·8	13·8 S	2	−5	(−31)	−2	0	6	6
Zikawei*.........	238·6	31·2 N	(−27)	−11	−6	(−20)	−7	8	......
Batavia	253·2	6·2 S	−6	−18	−6	−10	−4	(−24)	(−30
Toungoo	263·5	18·9 N	6	−3	−5	7	18	−3	−4
Barrackpore ...	271·6	22·8 ,,	3	−4	7	...	7	−3	1
Dehra Dun	281·9	30·3 ,,	2	−2	9	12	−3	4	−5
Kodaikanal	282·5	10·2 ,,	11	−2	5	18	(20)	−1	1
Bombay	287·2	18·9 ,,	−3	1	−4	1	−3	7	(
Ekaterinburg ..	299·4	56·8 ,,	0	7	9	10	6	(24)	(2
Mauritius	302·4	20·1 S	−7	19	−16	†	†	†	†
Helwan*	328·7	29·9 N	...	...	...	...	−17	8	(
Pola	346·2	44·9 ,,	9	(−21)	(−20)	−11	(24)	(−29)	−1
Potsdam	346·9	52·4 ,,	−2	−6	−6	−6	5	5	−
Rude Skov	347·5	55·8 ,,	...	...	...	...	8	......	
Munich	348·4	48·1 ,,	−4	14	0	−10	(−21)	?	?
Wilhelmshaven	351·9	53·5 ,,	−3	18	8	(−26)	−8	−18	(−1
De Bilt	354·8	52·1 ,,	−3	−5	...	...	...	−7	
Uccle	355·6	50·8 ,,	3	14	(23)	−14	−3	(36)	−
Val Joyeux......	358·0	48·8 ,,	−7	−12	−6	−16	3	(−46)	(−
Ebro	359·5	40·8 ,,	...	...	...	(39)	−2	(−28)	
Mean numerical residual			4_{25}	8_{23}	6_{20}	7_{20}	6_{26}	6_{21}	6
The unit in this part of the Table is 1 second (not 0·1 minute, as above).	Mean P.M. residual .		-7_{17}	4_{18}	3_{11}	-4_{13}	2_{18}	5_{8}	
	Mean A.M. residual .		6_{8}	-10_{5}	-5_{9}	-8_{7}	-8_{8}	1_{13}	−
	P.M.–A.M.		13	−14	−8	−4	−10	−4	−
	"Day" residuals ...		-10_{6}	24_{14}	13_{5}	-27_{11}	1_{16}	8_{11}	
	"Night" residuals .		-1_{19}	-35_{9}	-5_{15}	21_{9}	-4_{10}	-9_{7}	−
	"Night"–"Day" .		9	−59	−18	48	−5	−17	−1

* These observatories gave times in whole minutes only. For

† These times were in error by positive amounts varying from

‡ A small movement, of perhaps local character, nearly coincid

§ Several observatories gave data for two movements in this

one is here considered, which necessitates exclusion of d

To face p. 212.

mean time of commencement.								
9	10	11	12	13	14	15	Mean residual.	Mean numerical residual.
1908, Sep. 11. $7^h 20{\cdot}7^m$	1908, Sep. 11. $21^h 46{\cdot}6^m$	1908, Sep. 28. $8^h 41{\cdot}9^m$	1908, Sep. 29. $1^h 32{\cdot}1^m$	1909, May 14. $4^h 55{\cdot}0^m$	1909, Sep. 25. $8^h 38{\cdot}2^m$	1909, Sep. 25. $11^h 40{\cdot}7^m$		
−7	4	1	9	(30)	18	−7	3_{12}	7
(−27)	4	1	−1	(20)	0	3	2_{12}	3
(−29)	−14	−7	19	−7	§	...	-4_{8}	9
...	...	...	...	...	§	8	8_{1}	8
8	−11	1	−11	0	(−22)	−17	1_{11}	8
(−36)	(−22)	(−29)	(−34)	(−26)	2	6	0_{7}	8
−2	6	−11	−3	−8	4	1	-2_{14}	5
−6	1	−6	−9	14	8	5	2_{14}	5
4	11	−1	−18	(23)	5	7	1_{13}	6
1	(25?)	−3	−5	−3	−5	−2	1_{14}	4
3	2	10	...	(21)	16	3	4_{10}	6
9	−6	12	−1	−2	−3	−1	-5_{13}	8
−1	11	−6	−5	...	...	...	0_{11}	5
−17	−16	(−29)	9	−10	−12	−7	-6_{11}	10
−18	5	7	(27)	2	5	19	-2_{12}	8
...	...	...	...	...	−8	−5	0_{9}	7
...	11	−6	18	...	−4	−5	2_{12}	6
(21)	5	3	−2	...	−12	−2	1_{18}	5
(25)	(20)	3	4	...	−7	12	4_{11}	5
0	−12	−5	−9	−13	−2	15	-3_{15}	6
...	...	9	−3	16	6	−7	5_{11}	7
†	†	†	†	†	†	†	-1_{3}	14
13	(24)	(−29?)	9	−10	−2	−17	0_{9}	11
...	†	−8	−10	(−27)	−11	−13	-9_{8}	11
0	10	7	−1	8	−4	3	1_{15}	4
(−1?)	9	(15?)	10	...	−5	...	2_{6}	8
18	−6	9	3	?	3	−17	1_{11}	8
−8	−2	−5	−7	?	2	?	-1_{11}	8
5	−5	−9	−6	3	−2	3	-5_{12}	5
6	−17	7	0	−4	4	7	0_{13}	7
−7	−7	−8	14	16	11	4	0_{13}	10
−10	−11	...	9	...	−11	5	-3_{7}	7
7_{20}	8_{23}	6_{24}	8_{26}	8_{15}	6_{27}	7_{27}		
-24_{6}	-10_{16}	18_{9}	-41_{6}	1_{5}	-13_{10}	-10_{13}		
6_{14}	-2_{7}	-13_{15}	16_{20}	1_{10}	6_{17}	9_{14}		
30	8	−31	57	0	19	19		
-8_{13}	25_{6}	0_{17}	12_{7}	-18_{5}	-9_{20}	-2_{21}		
7_{7}	-19_{17}	-4_{7}	0_{19}	10_{10}	23_{7}	7_{6}		
15	−44	−4	−12	28	32	9		

e decimal 0 has been supposed added.
s.
scured the precise commencement in this case.
t one being at 8h. 30m. approximately. Only the later
vatories which measured only the former one.

electromagnetic waves to other parts. He was not aware of any direct evidence confirmatory of Dr. Chapman's statement that the atmospheric currents causing the diurnal variation were mainly situated in the comparatively near neighbourhood of the station. Also, even if this were the case, he did not see that any inference could be drawn as to what happened in the case of so different a phenomenon as a "sudden commencement." We did not even know whether the electrical currents causing the two phenomena were at the same level in the atmosphere. "Sudden commencements" were sometimes large, 50 γ or even 100 γ. They were not instantaneous changes, the rise normally seen in horizontal force in ordinary latitudes taking usually four or five minutes to attain its full value, a curious feature being that the high value was generally retained for some time, sometimes, in fact, for several hours, especially at stations in certain latitudes. A consideration of the exact nature of the phenomenon was a desirable prelude to any theorising. On the historical question which Dr. Chapman had raised, as to the discovery of a tendency to "repetition" in magnetic storms, it was very natural for him as a Greenwich man to emphasise the really valuable work which Mr. Maunder had done in this connection, but it was only fair to remember the much earlier work of Broun, and the independent work of A. Harvey and the late Prof. Birkeland. One reason why a belief in the "repetition" of magnetic storms had advanced so slowly was that sunspot theories had been so often suggested by cranks that they were naturally somewhat suspect. In the case of magnetic disturbances, and Mr. Maunder's Greenwich lists were no exception to the rule, the great majority did not have a "sudden commencement," and opinions as to when the average storm began or ended might differ by several hours. It was thus in general impossible to assign an exact value to the interval between two storms. Further, what one man called a "storm" another did not, so where one might see a "repetition" another would not. A further complication was the existence of a diurnal period in disturbance data, representing partly statistical imperfections and partly a real natural phenomenon. Dr. Chapman did not seem to realise that the 27-day period was just as manifest in quiet as in disturbed conditions. It was also important to remember that it had presented itself in years like 1913, when sunspots were almost non existent, just as decisively as in years of many sunspots.

Prof. Newall said that when Maunder's results were first brought forward it was felt to be a difficulty that they should involve an almost geometrical recurrence; because, of course, the disturbances on the surface of the sun were travelling with varying periods. When the slide showing the periodic recurrence of certain storms was first shown by Maunder it was noticed that a 1 per cent. difference in the adopted rotation period would change the direction of the lines which indicated coincidence of period to 45 deg. from the vertical. One saw on looking at the slide that there were many cases in which lines at different angles would suit. No one familiar with eclipse phenomena could doubt, however, that something originated in the sun. Many of the streamers sent out from the corona seemed quite straight, and it was difficult to resist the idea that sometimes the earth would be at the other end. Had Dr. Chapman considered the difficulty raised by Kelvin of the necessary energy relations? The energy of the magnetic storm itself might be attributed to something abstracted from the energy of rotation of the earth by the particles after entering the atmosphere.

Father Cortie thought Maunder was the first to connect the 27-day period with the solar streamers. There seemed to him to be a difficulty about these streamers of charged particles. Schuster showed that owing to the mutual repulsion of the charges the stream would be dispersed. He did not think that any particular storm was associated with

a particular sunspot, but with a disturbed area of the sun which might extend many degrees. His idea was that the dispersed streamers from the disturbed regions formed clouds of particles and that the earth may enter such a cloud, giving rise to magnetic storms. From the table it seemed to him that there was a slight preponderance of effect on the night side relative to Greenwich—*i.e.*, over the Pacific ocean. Would the presence of a large expanse of water be likely to affect the phenomena ?

Mr. T. Smith asked why the times of sudden commencement were chosen. He would imagine that similar time relationships would hold for any prominent feature. One would normally expect greater irregularities just at the beginning of the storm than when it was well under way. When it was known that a storm had commenced, wide scale runs could be started and measurements made on some subsequent outstanding feature.

A Member asked what irregularities might be due to lag in the different instruments used in the different observations.

Prof. Lees said that when Dr. Chree first brought the matter before the Society he had set some of his students to work out the correlation factor of Bauer's figures. The factor was so low that there was evidently no basis whatever for Bauer's theory.

Dr. Chapman, in reply, said he did not put any reliance on the actual figures given in the Paper. The observational error was too great. It was the method of treating the observations that he thought was of importance. He thought diurnal variations were produced at lower levels of the atmosphere than storms. By "near neighbourhood" he meant within about 1,000 kilometres or so. He did not think Maunder connected the storms with actual sunspots, but rather with disturbed regions. Of course, any proper motion of the disturbed region on the sun would alter the period in particular cases ; but in examining a long series it was best to take the synodic period since the others would be distributed on either side of this. He had not yet gone into the energy question, but did not think that this would present any insuperable difficulties. He did not think the streams would diffuse very much. He thought they consisted of α particles. They did not necessarily proceed radially from the sun, but might emerge in all directions from radial to tangential. This, however, would not seriously affect the time taken to pass across the earth. There was a definite reason for chosing the commencement times of the storms. When the storm is under way the magnetic state of the earth is fluctuating, and it is difficult to recognise accurately particular features. There was certainly a lag in the measurements which differed in different instruments ; but the effect of this was greatly reduced in the present method of grouping the observations, since the same instrument on different occasions contributes results to different groups. In this respect the method has a great advantage over classification on a geographical basis, in which case any instrument is always in the same group.

XVII. *The Entropy of a Metal.*

By H. STANLEY ALLEN, M.A., D.Sc., *University of London, King's College.*

RECEIVED APRIL 12TH, 1918.

§ 1. According to the theory of Ratnowsky* the entropy of a gram atom of a substance in the solid state may be expressed in the form,

$$S=\frac{3Nk}{x^3}\left\{4\int_0^x\frac{\xi^3 d\xi}{e^\xi-1}-x^3\log\left(1-e^{-x}\right)\right\},$$

where N is Avogadro's constant, k is the gas constant for a single molecule, and $x=h\nu_m/kT=\beta\nu/T=\Theta/T$, h being Planck's constant, ν_m the maximum vibration frequency of Debye's theory, and Θ a temperature characteristic of the substance considered.

In a previous communication † I have shown that the correct form of the approximation for small values of x, corresponding to high values of the absolute temperature T, is

$$S=3Nk\left\{\frac{4}{3}-\log x+\frac{x^2}{40}\cdots\right\},$$

where $3Nk=C_\infty=5{\cdot}96$ calories.

§ 2. Up to the present time no direct evidence as to the validity of this equation has been brought forward, but an important Paper by Lewis and Gibson ‡ on the entropy of the elements now provides the data required for testing the formula. The work of these authors is based on the heat theorem of Nernst—the so-called third law of thermodynamics —which may be stated in the form: "In an isothermal process involving pure solids and liquids the change in entropy approaches zero as the temperature approaches the absolute zero." Following Planck, the principle may be re-stated in the more general form: "The entropy of every actual substance in the pure state is zero at the absolute zero of temperature." Hence

$$S=\int_0^T\frac{dQ}{T}=\int_0^T\frac{CdT}{T},$$

* Ratnowsky, "Deutsch. Physikal. Gesell. Verh." Vol. XVI., p. 1033· 1914.

† H. S. Allen, "Proc. Phys. Soc." Vol. XXVIII., p. 302, 1916.

‡ Lewis and Gibson, "Am. Chem. Soc. Journ." Vol. XXXIX., p. 2554, 1917.

where C is the atomic heat. The absolute value of the entropy may be calculated, either at constant volume or at constant pressure, for any assigned temperature, provided C is known as a function of the temperature. The primary object of Lewis and Gibson was to determine the entropies of the elements in their standard states, at the standard temperature of free-energy measurements, namely 25°C. or 298°K. For many substances the atomic heat at constant volume is given by the equation

$$C_v = f(T/\theta),$$

where f is the same function for different substances, and θ a characteristic constant for each substance. Instead of making any assumption as to the exact form of the heat-capacity equation, the authors employed a graphic method of calculating the entropy of solid substances for which the equation has the same form. It is to be noted that θ is defined by them as the temperature at which C_v is one-half of the Dulong and Petit constant, so that when $T=\theta$, $C_v=3R/2=\frac{1}{2}\,C_\infty$.

§ 3. The following table contains the values of the characteristic constants and the entropies for seven metals which have been made the subject of accurate investigations at low temperatures. The column headed θ gives the characteristic temperature as defined by Lewis and Gibson, Θ is the characteristic temperature employed by Debye.* The entropy in calories per degree at 298° K., under the condition of constant volume, is given in the last two columns of the table. The observed values are quoted from the Paper by Lewis and Gibson,† whilst the calculated values have been obtained from the approximate form of the formula of Ratnowsky. The agreement between the observed and the calculated

* It is noteworthy that the values of Θ in the table are almost exactly four times the corresponding values of θ. It is interesting to test this relationship assuming Debye's expression for the specific heat, which may be written

$$\frac{C_v}{C_\infty}=\frac{12}{x^3}\int_0^x\frac{\xi^3 d\xi}{e^\xi-1}-\frac{3x}{e^x-1}.$$

I have to thank Prof. J. B. Dale for calculating the value of the right hand side of this equation when $x=4$, that is when $\Theta=4T$. It is found that $C_v/C_\infty=0{\cdot}503059$, which differs from one-half by less than one per cent. This result shows that when Debye's theory is employed, the characteristic temperature, Θ, is almost, but not exactly, equal to four times the temperature at which C_v is one-half of the limiting value C_∞.

† These authors also give values for the entropy of the elements under constant pressure.

values is remarkably good. The letter *a* after an observed value indicates that the value in question is hardly likely to be in error by more than one-third of an entropy unit.

ENTROPY OF THE METALS AT 298° K.

Element.	θ.	Θ.	S_c, observed.	S_c, calculated.
Aluminium	95·5	382 S	6·73 *a*	6·72
Iron	99·1	385 G	6·54	6·68
Copper	78·2	315 K	7·91 *a*	7·79
Zinc	57·6	230 G, N	9·60 *a*	9·59
Silver	53·7	215 G	10·00 *a*	9·98
Cadmium	42·4	168 G	11·38	11·42
Lead	22·0	88 K, S	15·11 *a*	14·89

G=Griffiths, K=Keesom and Onnes, N=Nernst, S=Schwers.

The thermal behaviour of sodium * is somewhat exceptional, and for that reason it has not been included in the table. At low temperatures Messrs. Griffiths find $\Theta=180$, which yields as the calculated value of the entropy 12·21 calories per degree, whilst the value deduced by Lewis and Gibson from Dewar's measurements is 11·43.

Another case in which it is possible to institute a comparison between the theoretical and the experimental value for the entropy is that of solid mercury at the temperature of the melting point. Nernst and Lindemann found $\Theta=96{\cdot}6$, so that at the melting point ($T=234{\cdot}1°$ K.) the entropy is 13·26 calories per degree. The corresponding value deduced by Lewis and Gibson from the observations on the specific heat is 13·31 calories per degree. Here the agreement could scarcely be improved upon.

§ 4. When x is not too large, the value of the entropy can be calculated from the formula of Ratnowsky expanded in the form

$$S=C_\infty\left\{\frac{4}{3}-\log x+\frac{x^2}{40}-\frac{x^4}{2240}+\frac{x^6}{108864}-\dots\right\}.$$

The series is convergent when x is (numeraiclly) less than 2π, but converges slowly when x is as large as 4. For large values of x the method of expansion given by Debye † may be employed, or by combining the expression of Ratnowsky for

* E. H. Griffiths and E. Griffiths, "Phil. Trans," A. Vol. CCXIV, p. 319, 1914. Eastman and Rodebush, "Journ. Am. Chem. Soc." Vol. XL., p. 489, 1918.

† Debye, "Annalen der Physik." Vol. XXXIX., p. 789, 1912.

the entropy with that of Debye for the specific heat the entropy may be found from the formula,

$$S=\tfrac{1}{3}C_v+C^\infty\left[\frac{x}{e^x-1}-\log\left(1-e^{-x}\right)\right],$$

in which the values for C_v calculated by Debye may be inserted.

In the diagram (Fig. 1) the value of the entropy in calories per degree, calculated as described above, has been plotted against the value of $1/x=T/\Theta$. For any assigned value of Θ this gives an entropy temperature diagram. In the same figure are given the points (marked $\times$) determined from the values of S_v recorded by Lewis and Gibson as corresponding to selected values of θ/T. In plotting these points it has been

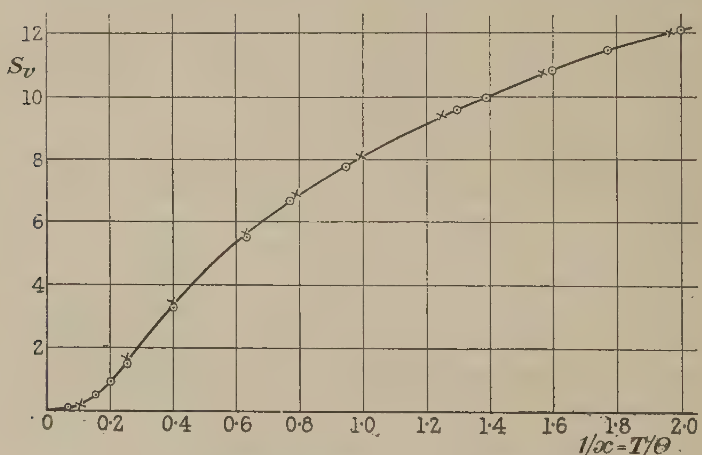


FIG. 1.—VARIATION OF ENTROPY WITH TEMPERATURE.
⊙ Formula of Ratnowsky. × Lewis and Gibson.

assumed that $\Theta=4{\cdot}021\theta$, which is the relation given by Debye as corresponding to $C_v=\tfrac{1}{2}C_\infty$. It will be seen that the two sets of points lie on a smooth curve, showing the close agreement between the theoretical and the experimental results throughout the whole range of temperatures considered.

§ 5. The above results prove that the formula of Ratnowsky gives values for the entropy of a solid in very close agreement with those obtained directly from observations on the specific

heat at various temperatures. It therefore becomes a matter of special interest to examine the principal assumptions made in the deduction of the formula. These appear to be four in number.

1. The whole internal energy, U, of the gram atom is made up of two parts, the "vibrational energy," E, and the potential energy $f(v)$, which is a function of the volume. Even at the absolute zero of temperature the substance possesses a certain amount of potential energy.

2. Boltzmann's equation for the entropy is assumed, so that

$$S=k \log W+C,$$

where W is the number of "complexions" of the system when it has a given energy. The constant of integration, C, is ignored or included as a constant multiplier in the quantity W, so that the equation reduces to

$$S=k \log W.$$

This, as Planck points out ("Heat Radiation," § 120), is equivalent to assigning a definite absolute value to the entropy S, and leads necessarily to Nernst's heat theorem.

3. It is assumed that

$$\frac{\partial S}{\partial E}=\frac{1}{T}.$$

This implies a differentiation at constant volume.

4. Following Debye, it is assumed that the number of vibrations within a frequency interval $d\nu$ is given by

$$9N\nu^2 d\nu/\nu^3{}_m,$$

where ν_m denotes the maximum vibration frequency. It is known that Debye's theory must be regarded as giving only a first approximation, since the existence of a sharply defined maximum frequency is a somewhat arbitrary assumption. Nevertheless the formula deduced for the atomic heat gives a very close approximation to the experimental results.

§ 6. We may conclude that the hypotheses assumed by Ratnowsky in dealing with the entropy of a solid are probably justified as being at least approximately true. It is necessary

to point out, however, that the results of the present inquiry have no bearing on the further assumption made by Ratnowsky that his expression for the entropy may be applied to the liquid state. This is a question which I have discussed in a former Paper dealing with the latent heat of fusion of a metal (*loc. cit.*), but it cannot be definitely decided till further information is available as to the applicability of Debye's theory to liquids.

ABSTRACT.

An expression for the entropy of one gram atom of a substance in the solid state has been given by Ratnowsky. In a communication to the Physical Society in 1916 the author gave the correct form of the approximation required for high values of the absolute temperature in terms of Bernoulli's numbers. The data required for testing the formula have been supplied in a recent Paper by Lewis and Gibson, who have given values for the entropy of the elements under the condition of constant volume, and also under constant pressure. These values were deduced from observations on the specific heat assuming the truth of the heat theorem of Nernst, that the entropy of every actual substance in the pure state is zero at the absolute zero of temperature. It is found that the formula of Ratnowsky gives values for the entropy of a solid in very close agreement with those obtained by Lewis and Gibson. The hypotheses assumed in the theory of Ratnoswky are discussed, and the conclusion is drawn that these are probably justified as being at least approximately true.

XVIII. *On Tracing Rays Through an Optical System. (Second Paper.)* *By* T. SMITH, *B.A.* (*From the National Physical Laboratory.*)

RECEIVED APRIL 22, 1918.

IN the earlier communication* bearing this title the author gave formulæ by which a ray in three dimensions could be traced through a system of coaxial spherical refracting surfaces. These formulæ are algebraic, and do not require the use of any tables; in this respect they are in marked contrast to those generally used. The new formulæ are primarily intended for use in conjunction with a calculating machine; the older trigonometrical formulæ have been arranged for logarithmic work. Experience has shown that logarithmic calculations take several times as long as the corresponding mechanical operations.

The present Paper contains the modification of these formulæ generally used for rays in an axial plane, and, for both two and three dimensional cases, new formulæ which are universally applicable, so that the calculations could, if required, be carried out entirely by mechanical means. In the three dimensional case a modification is only required for the preliminary calculation described in the previous Paper. The present Paper also includes the method followed in calculations relating to transverse focal lines.

By far the greater number of rays that have to be traced through systems of lenses lie in a plane containing the axis. The system used in the previous Paper can be modified for these cases. Let φ_λ, φ'_λ be the angles of incidence and emergence at the surface which separates media of refractive indices $\mu_{\lambda-1}$ and μ_λ. The inclination to the axis of the ray between surfaces λ and $\lambda+1$ is ψ_λ; t_λ and a_λ† are the distances between the vertices and the centres of curvature respectively of these surfaces. The radius of curvature of the surface λ is r_λ, and its curvature is $R_\lambda=1/r_\lambda$. The length of the perpendicular from the centre of curvature of this surface on to the incident ray is h_λ†, and on to the refracted ray is h'_λ. To find the position of the final emergent ray the formulæ

* Proc. Phys. Soc., Vol. XXVII., page 502.

† There are minor alterations from the notation employed in the previous Paper.

$$h_\lambda = h'_{\lambda-1} + a_{\lambda-1} \sin \psi_{\lambda-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$\mu_\lambda h'_\lambda = \mu_{\lambda-1} h_{\lambda-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$\sin \varphi_\lambda = h_\lambda R_\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$
$$\sin \varphi'_\lambda = h'_\lambda R_\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$
$$\psi_\lambda = \varphi'_\lambda - \varphi_\lambda + \psi_{\lambda+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$
$$a_\lambda = t_\lambda - r_\lambda + r_{\lambda-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

are successively applied at each refracting surface.

Of these formulæ, (1), (2), (3), (4), (6) are equivalent to some of the general formulæ contained in the previous Paper. Equation (5) constitutes a modification which applies only to the particular case of rays in two dimensions. The formulæ as given above involve reference to tables of natural sines, since equation (5) refers to angles, and not to their trigonometrical functions. On paper, owing to the algebraic simplicity of equation (5), the system appears to be less complex than before; in fact, the simplification is illusory, for it actually involves at least three references to tables for each surface, and these take up most of the time and also involve most chance of error. The system as given above is the form that is generally used; but experience has shown that it is quite as quick to avoid reference to tables at all. The calculations are generally arranged in three parallel columns as follows:—

$$\begin{tikzcd}
\text{Angle.} & \text{Sine.} & \text{Central perpendicular.} \\
\psi_0 \arrow[r] & \sin \psi_0 & {} \arrow[d] \\
\varphi_1 & \sin \varphi_1 \arrow[l] & h_1 \arrow[l] \arrow[d] \\
\varphi_1' \arrow[d] & \sin \varphi_1' \arrow[l] & h_1' \arrow[l] \\
\psi_1 \arrow[r] & \sin \psi_1 \arrow[r] & h_2 - h_1' \arrow[d] \\
\varphi_2 & \sin \varphi_2 \arrow[l] & h_2 \arrow[l] \arrow[d] \\
\varphi_2' \arrow[d] & \sin \varphi_2' \arrow[l] & h_2' \arrow[l] \\
\psi_2 \arrow[r] & \sin \psi_2 \arrow[r] & h_3 - h_2' \arrow[d] \\
 & & {}
\end{tikzcd}$$

The arrows show the order in which the operations are carried out.

It frequently happens that, after the ray has been traced, it is required to find the positions at which it is intersected by the focal lines formed by a pencil of neighbouring rays. The calculation of the radial focal line was considered in the previous Paper. It is most simply effected by finding the length of the ray intercepted between successive surfaces and the power of the various surfaces for that ray. The formula for the intercepted length d_λ corresponding to the axial length t_λ is

$$d_\lambda = a_\lambda \cos \psi_\lambda + r_\lambda \cos \varphi'_\lambda - r_{\lambda+1} \cos \varphi_{\lambda+1} \quad . \quad . \quad (7)$$

and the power of the surface for this ray, K_λ, is given by

$$K_\lambda = (\mu_\lambda \cos \varphi'_\lambda - \mu_{\lambda-1} \cos \varphi_\lambda) R_\lambda \quad . \quad . \quad . \quad . \quad . \quad (8)$$

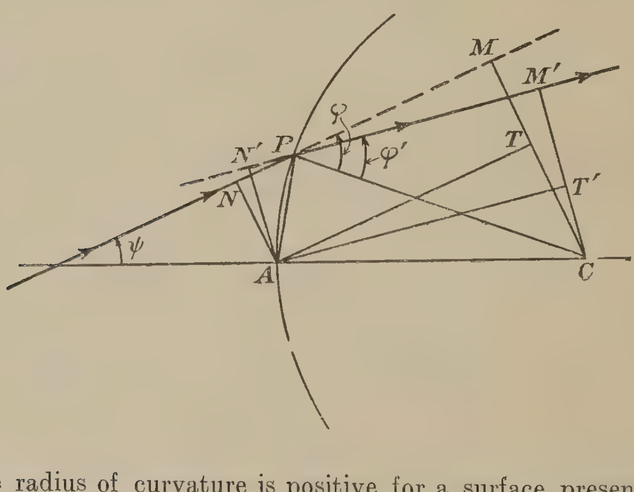
The position of the radial focal line is found by regarding d_λ and K_λ as t_λ and κ_λ, and applying the formulæ for paraxial rays to find conjugate points. The determination of the transverse focal line is considered later.

With the above treatment, as with all other systems that the author has met, special formulæ are necessary when flat or nearly flat surfaces are encountered. The application of the formulæ as they stand to nearly flat surfaces would result in considerable loss of accuracy, because two successive a's are very great, but of opposite sign, and with a given number of figures in $\sin \psi$ the sum of the two successive products $a \sin \psi$ is uncertain to an extent which increases with the magnitude of the radius. To avoid this uncertainty, and yet be able to treat all surfaces alike, whether they are flat, nearly flat or very considerably curved, an alternative method of calculation has been devised.

Consider for a moment the conditions which such a system must satisfy. In the first place, all points of reference must be at a finite distance from the portions of the surfaces operative in producing refraction; thus, no reference is possible to the centre of curvature or to the point of intersection of a ray with the axis, since either may be at infinity. Again, the radius of curvature may not be used, for this may become infinite; on the other hand, its reciprocal, the curvature, may be employed, since it is always finite or zero. More generally, no lengths measured along the axis may be used if high accuracy is desired, because these are so variable in magnitude. Transverse distances, on the other hand, vary within small limits fixed

by the apertures of the various lenses, and their use will tend to give uniform reliability at all surfaces. Lastly, if the formulæ involve fractions, the denominators must in all cases be essentially constant in sign and large in magnitude. It will be shown that all these conditions can be satisfied.

Take, first, a ray which lies in an axial plane. The figure shows the incident ray NP refracted at P as the emergent ray PM'. Let C be the centre of curvature of the refracting surface, and A the point at which the axis of the system meets the surface. Let M, M' be the feet of the perpendiculars to the incident and refracted rays from the centre of curvature C, and N, N' the feet of those from the vertex A. The diagram shows the case in which the various quantities entering the formulæ are positive. Light travels from left to right, and



the radius of curvature is positive for a surface presenting its convex side to the incident light. The inclinations ψ, ψ' of the rays to the axis are positive when the rotation of a straight line from the direction of the axis through an acute angle in the counter-clockwise direction will bring it into parallelism with the ray. Similarly, the angles φ, φ' of incidence and emergence are positive when a rotation through an acute angle in the counterclockwise direction from the direction of the normal to the surface at the point of refraction brings the line into coincidence with the direction of the ray. AT and AT' are perpendiculars from A to CM and CM'. The radius and the central perpendiculars will be denoted as before by r,

h and h', and the perpendiculars AN, AN' by $\mathfrak{h}$ and $\mathfrak{h}'$. The chord AP may be denoted by $\mathfrak{H}$. From the figure

$$CM = r \sin \varphi,$$
$$CT = r \sin \psi,$$
$$AN = CM - CT,$$

or

$$\mathfrak{h} = r \sin \varphi - r \sin \psi.$$

In this form the equation may not be used, because it involves r. Dividing throughout by r and rearranging, the equation becomes

$$\sin \varphi = \sin \psi + \mathfrak{h} R \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

a form which is free from objection. If, then, the incident ray is given by ψ and $\mathfrak{h}$, equation (9) enables the angle of incidence to be found. The angle of emergence is given by the law of refraction

$$\mu' \sin \varphi' = \mu \sin \varphi$$

and the value of ψ' by

$$\psi' = \varphi' - \varphi + \psi.$$

At the next surface the length of the perpendicular from the vertex to the incident ray is evidently

$$\mathfrak{h}' + t \sin \psi',$$

where t is the axial distance between the two vertices. It therefore only remains to determine the value of $\mathfrak{h}'$. This may not be found from the equation,

$$\mathfrak{h}' = AN' = CM' - CT' = r(\sin \varphi' - \sin \psi'),$$

since this involves r. Neither may r be eliminated by (9), because the denominator in

$$\mathfrak{h}' = \mathfrak{h} \frac{\sin \varphi' - \sin \psi'}{\sin \varphi - \sin \psi}$$

becomes very small for long radii. Since, however, $\varphi' - \psi' = \varphi - \psi$, this equation may be written

$$\frac{\mathfrak{h}'}{\mathfrak{h}} = \frac{\cos \frac{1}{2}(\varphi' + \psi')}{\cos \frac{1}{2}(\varphi + \psi)} = \frac{\cos \varphi' + \cos \psi'}{\cos \varphi + \cos \psi},$$

and in this form the conditions laid down above are satisfied.* As the cosines of φ, φ', ψ and ψ' are required in dealing with the refraction of neighbouring rays, these are the quantities

* It may be noted that $\mathfrak{h} / \cos \frac{1}{2}(\varphi + \psi) = \mathfrak{h}' / \cos \frac{1}{2}(\varphi' + \psi') = \mathfrak{H}$, since the angles PAN, PAN' are equal to $\frac{1}{2}(\varphi + \psi)$ and $\frac{1}{2}(\varphi' + \psi')$ respectively.

that will normally be used. The equations of the new system are thus

$$\sin \varphi_\lambda = \sin \psi_{\lambda-1} + \mathfrak{h}_\lambda R_\lambda . \quad (9)$$

$$\mu_\lambda \sin \varphi'_\lambda = \mu_{\lambda-1} \sin \varphi_\lambda \quad (10)$$

$$\psi_\lambda = \varphi'_\lambda - \varphi_\lambda + \psi_{\lambda-1} \quad (5)$$

$$\mathfrak{h}'_\lambda = \mathfrak{h}_\lambda \frac{\cos \varphi'_\lambda + \cos \psi_\lambda}{\cos \varphi_\lambda + \cos \psi_{\lambda-1}} \quad (11)$$

$$\mathfrak{h}_{\lambda+1} = \mathfrak{h}'_\lambda + t_\lambda \sin \psi_\lambda \quad (12)$$

The calculations are arranged very similarly to those set out diagrammatically above. Four columns are used—for the angles, their sines, their cosines and the perpendiculars from the vertices. It may be noted that the quantities found suffice to check all the calculations, since both the sines and the cosines of all the angles are known. In particular, absolute checks are available for all the quantities abstracted from the tables.

The power of the surface λ for the ray considered is derived as before from (8), but equation (7) has to be replaced. Several formulæ are available. If $\mathfrak{H}$ has been determined for the various surfaces, projection upon the axis gives

$$d_\lambda \cos \psi_\lambda = t_\lambda - \tfrac{1}{2}\mathfrak{H}^2_\lambda R_\lambda + \tfrac{1}{2}\mathfrak{H}^2_{\lambda+1} R_{\lambda+1} \quad (13)$$

or projection upon the ray leads to

$$d_\lambda = t_\lambda \cos \psi_\lambda - \sqrt{(\mathfrak{H}^2_\lambda - \mathfrak{h}'^2_\lambda)} + \sqrt{(\mathfrak{H}^2_{\lambda+1} - \mathfrak{h}^2_{\lambda+1})} . \quad (14)$$

When $\mathfrak{H}$ is unknown, perhaps the simplest formula which does not involve the determination of any new quantities is

$$d_\lambda = t_\lambda \cos \psi_\lambda - \mathfrak{h}'_\lambda \frac{\sin \varphi'_\lambda + \sin \psi_\lambda}{\cos \varphi'_\lambda + \cos \psi_\lambda} + \mathfrak{h}_{\lambda+1} \frac{\sin \varphi_{\lambda+1} + \sin \psi_\lambda}{\cos \varphi_{\lambda+1} + \cos \psi_\lambda} \quad (15)$$

Other interesting formulæ are obtained by projecting on to a line bisecting the angle between the ray and the axis. A simple formula of this class is

$$d_\lambda - t_\lambda = (\mathfrak{H}_{\lambda+1} \sin \tfrac{1}{2}\varphi_{\lambda+1} - \mathfrak{H}_\lambda \sin \tfrac{1}{2}\varphi'_\lambda) \sec \tfrac{1}{2}\psi_\lambda .$$

From this it follows that the optical length of the ray between the feet of perpendiculars to the incident and final emergent

paths from the two external vertices exceeds the axial path between these vertices by

$$\tfrac{1}{2}\Sigma \mathfrak{H}_\lambda \mu_\lambda \sin \varphi'_\lambda \{\sec \tfrac{1}{2}\psi_{\lambda-1} \sec \tfrac{1}{2}\varphi_\lambda - \sec \tfrac{1}{2}\psi_\lambda \sec \tfrac{1}{2}\varphi'_\lambda\}$$

$$= \tfrac{1}{4}\Sigma \frac{\mathfrak{H}_\lambda \mu_\lambda \sin \varphi_\lambda'}{\cos \frac{1}{2}\psi_{\lambda-1} \cos \frac{1}{2}\psi_\lambda \cos \frac{1}{2}\varphi_\lambda \cos \frac{1}{2}\varphi'_\lambda} \{\cos \tfrac{1}{2}(\varphi'_\lambda + \psi_\lambda) - \cos \tfrac{1}{2}(\varphi_\lambda + \psi_{\lambda-1})\}$$

$$= \tfrac{1}{4}\Sigma \frac{(\mathfrak{h}'_\lambda - \mathfrak{h}_\lambda)\mu_\lambda \sin \varphi'_\lambda}{\cos \frac{1}{2}\psi_{\lambda-1} \cos \frac{1}{2}\psi_\lambda \cos \frac{1}{2}\varphi_\lambda \cos \frac{1}{2}\varphi'_\lambda}$$

$$= \Sigma \frac{(\mathfrak{h}'_\lambda - \mathfrak{h}_\lambda)\mu_\lambda \sin \varphi'_\lambda}{\sqrt{\{(1+\cos \psi_{\lambda-1})(1+\cos \psi_\lambda)(1+\cos \varphi_\lambda)(1+\cos \varphi'_\lambda)\}}}. \qquad (16)$$

A number of other forms can be found for this path difference, but it is unnecessary to pursue the subject further here.

Before leaving the consideration of a ray lying in an axial plane, it may be noted that the two series of formulæ using the centres of curvature and the vertices of the surfaces respectively as reference points may be used in conjunction with one another. Reference to the centres of curvature would be made for short radii, and to the vertices for long radii. All the formulæ remain as before, except that, on changing from one system to the other, instead of a_λ or t_λ, the axial distance between the two reference points is used. For example, if the centre of curvature is the reference point for surface λ and the vertex for surface $\lambda+1$, equation (1) will be replaced by

$$\mathfrak{h}_{\lambda+1} = h'_\lambda + b_\lambda \sin \psi_\lambda,$$

where $$b_\lambda = t_\lambda - r_\lambda.$$

If at surface $\lambda+2$ the centre of curvature is employed again, the transition equation will be

$$h_{\lambda+2} = \mathfrak{h}'_{\lambda+1} + c_{\lambda+1} \sin \psi_{\lambda+1},$$

where $$c_{\lambda+1} = t_{\lambda+1} + r_{\lambda+2}.$$

The corresponding changes in the formulæ for d are obvious, and need not be set down here.

The extension of the formulæ to rays which do not always lie in the same plane may now be considered. As in the earlier Paper, let the ray incident in the medium $\mu_{\lambda-1}$ on the surface λ be determined in direction by the direction cosines $L_{\lambda-1}$, $M_{\lambda-1}$, $N_{\lambda-1}$; and the refracted ray in the medium μ_λ by L_λ, M_λ, N_λ. The direction cosines of the normal to the surface at the point

where refraction takes place are l_λ, m_λ, n_λ. The previous notation involving φ and ψ may be retained also, so that

$$L_\lambda = \cos \psi_\lambda$$

and
$$L_{\lambda-1} l_\lambda + M_{\lambda-1} m_\lambda + N_{\lambda-1} n_\lambda = \cos \varphi_\lambda$$
$$L_\lambda l_\lambda + M_\lambda m_\lambda + N_\lambda n_\lambda = \cos \varphi'_\lambda$$

In the two dimensional case the perpendicular from the vertex to the incident ray made angles $\frac{\pi}{2} + \psi_{\lambda-1}$ and $\frac{\pi}{2} + \varphi_\lambda$ with the axis and the normal to the surface. In the general case this will no longer be true. Let the direction cosines of the two perpendiculars from the vertex to the incident and emergent rays be

$$e_\lambda, \ f_\lambda, \ g_\lambda$$

and
$$e'_\lambda, \ f'_\lambda, \ g'_\lambda$$

respectively. The co-ordinates of the feet of the perpendiculars are thus

$$e_\lambda \mathfrak{h}_\lambda, \ f_\lambda \mathfrak{h}_\lambda, \ g_\lambda \mathfrak{h}_\lambda$$

and
$$e'_\lambda \mathfrak{h}'_\lambda, \ f'_\lambda \mathfrak{h}'_\lambda, \ g'_\lambda \mathfrak{h}'_\lambda$$

where the vertex of the λth surface is taken as origin. The point of refraction is

$$r_\lambda(1 - l_\lambda), \ -r_\lambda m_\lambda, \ -r_\lambda n_\lambda,$$

and the lengths intercepted on the rays between the perpendiculars and this point are $\sqrt{(\mathfrak{H}^2_\lambda - \mathfrak{h}^2_\lambda)}$ and $\sqrt{(\mathfrak{H}^2_\lambda - \mathfrak{h}'^2_\lambda)}$. Therefore

$$\sqrt{(\mathfrak{H}^2_\lambda - \mathfrak{h}^2_\lambda)} = \frac{r_\lambda(1 - l_\lambda) - e_\lambda \mathfrak{h}_\lambda}{L_{\lambda-1}} = -\frac{r_\lambda m_\lambda + f_\lambda \mathfrak{h}_\lambda}{M_{\lambda-1}} = -\frac{r_\lambda m_\lambda + g_\lambda \mathfrak{h}_\lambda}{N_{\lambda-1}} \qquad (17)$$

Multiply the fractions, both numerator and denominator, by e_λ, f_λ, g_λ and add. Since

$$e_\lambda L_{\lambda-1} + f_\lambda M_{\lambda-1} + g_\lambda N_{\lambda-1} = 0,$$

it follows that

$$r_\lambda \{e_\lambda - (e_\lambda l_\lambda + f_\lambda m_\lambda + g_\lambda n_\lambda)\} = \mathfrak{h}_\lambda$$

or
$$e_\lambda l_\lambda + f_\lambda m_\lambda + g_\lambda n_\lambda = e_\lambda - \mathfrak{h}_\lambda R_\lambda \quad . \quad . \quad . \quad . \quad (18)$$

the equation which leads to the value of the angle of incidence in the two dimensional case. Again multiply the fractions by $1 + l_\lambda$, m_λ, n_λ, and add as before; this gives

$$\sqrt{(\mathfrak{H}^2_\lambda - \mathfrak{h}^2_\lambda)} = -(e_\lambda + e_\lambda l_\lambda + f_\lambda m_\lambda + g_\lambda n_\lambda)\mathfrak{h}_\lambda / (\cos \psi_{\lambda-1} + \cos \varphi_\lambda)$$
$$= (\mathfrak{h}_\lambda{}^2 R_\lambda - 2 e_\lambda \mathfrak{h}_\lambda)/(\cos \psi_{\lambda-1} + \cos \varphi_\lambda) \quad . \quad . \quad (19)$$

by (18). Also multiplying by $L_{\lambda-1}$, $M_{\lambda-1}$, $N_{\lambda-1}$ and adding

$$\sqrt{(\mathfrak{H}_\lambda - \mathfrak{h}^2_\lambda)} = r_\lambda(\cos \psi_{\lambda-1} - \cos \varphi_\lambda). \quad . \quad . \quad . \quad (20)$$

The elimination of $\sqrt{(\mathfrak{H}_\lambda^2 - \mathfrak{h}_\lambda^2)}$ between (19) and (20) leads to

$$\cos^2 \varphi_\lambda = \cos^2 \psi_{\lambda-1} + 2e_\lambda \mathfrak{h}_\lambda R_\lambda - \mathfrak{h}_\lambda^2 R_\lambda^2 \quad . \quad . \quad . \quad . \quad (21)$$

as an equation for the determination of φ_λ. The value of φ'_λ will be derived from

$$\mu_\lambda \sin \varphi'_\lambda = \mu_{\lambda-1} \sin \varphi_\lambda$$

and K_λ from

$$K_\lambda = (\mu_\lambda \cos \varphi'_\lambda - \mu_{\lambda-1} \cos \varphi_\lambda) R_\lambda.$$

When φ_λ has been found $\mathfrak{H}_\lambda^2$ is derived from (19). Equation (20) cannot, of course, be used for this purpose. It is next necessary to determine ψ_λ. This could be derived directly from the law of refraction

$$\mu_\lambda \cos \varphi'_\lambda - \mu_{\lambda-1} \cos \varphi_\lambda = \frac{\mu_\lambda \cos \psi_\lambda - \mu_{\lambda-1} \cos \psi_{\lambda-1}}{l_\lambda}$$

$$= \frac{\mu_\lambda \cos \psi_\lambda - \mu_{\lambda-1} \cos \psi_{\lambda-1}}{1 - \frac{1}{2} \mathfrak{H}_\lambda^2 R_\lambda^2}$$

but it is preferable to find $\sqrt{(\mathfrak{H}_\lambda^2 - \mathfrak{h}'^2_\lambda)}$ directly, and thence obtain ψ_λ, using a relation which may be written down by analogy with (20)—viz.,

$$\cos \psi_\lambda = \cos \varphi'_\lambda + R_\lambda \sqrt{(\mathfrak{H}_\lambda^2 - \mathfrak{h}'^2_\lambda)}. \quad . \quad . \quad . \quad . \quad (22)$$

The reason for this procedure is of course that, under the conditions which have been laid down, $\cos \psi_\lambda$ can be found from the value of $\sqrt{(\mathfrak{H}_\lambda^2 - \mathfrak{h}'^2_\lambda)}$; but the process may not be reversed. To obtain an expression for $\sqrt{(\mathfrak{H}_\lambda^2 - \mathfrak{h}'^2_\lambda)}$, write the equation of refraction in the form

$$\mu_\lambda r_\lambda (\cos \psi_\lambda - \cos \varphi'_\lambda) = \mu_{\lambda-1} r_\lambda (\cos \psi_{\lambda-1} - \cos \varphi_\lambda)$$
$$- (1 - l_\lambda) r_\lambda (\mu_\lambda \cos \varphi'_\lambda - \mu_{\lambda-1} \cos \varphi_\lambda)$$

or $$\mu_\lambda \sqrt{(\mathfrak{H}_\lambda^2 - \mathfrak{h}'^2_\lambda)} = \mu_{\lambda-1} \sqrt{(\mathfrak{H}_\lambda^2 - \mathfrak{h}_\lambda^2)} - \tfrac{1}{2} \mathfrak{H}_\lambda^2 K_\lambda. \quad . \quad . \quad (23)$$

The equation

$$e'_\lambda \mathfrak{h}'_\lambda = \tfrac{1}{2} \mathfrak{H}_\lambda^2 R_\lambda - \cos \psi_\lambda \sqrt{(\mathfrak{H}_\lambda^2 - \mathfrak{h}'^2_\lambda)} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

derived from a relation similar to (17), completes the determination of the three quantities $\mathfrak{h}'^2_\lambda$, $e'_\lambda \mathfrak{h}'_\lambda$ and ψ_λ from the given values of $\mathfrak{h}_\lambda^2$, $e_\lambda \mathfrak{h}_\lambda$ and $\psi_{\lambda-1}$.

The length on the ray in medium μ_λ between the perpen-

diculars from the vertices of the surfaces λ and $\lambda+1$ is evidently $t_\lambda \cos \psi_\lambda$, and therefore

$$t_\lambda \cos \psi_\lambda = \frac{e_{\lambda+1}\, \mathfrak{h}_{\lambda+1} - e'_\lambda\, \mathfrak{h}'_\lambda + t_\lambda}{L_\lambda} = \frac{f_{\lambda+1}\, \mathfrak{h}_{\lambda+1} - f'_\lambda\, \mathfrak{h}_\lambda}{M_\lambda}$$

$$= \frac{g_{\lambda+1}\, \mathfrak{h}_{\lambda+1} - g'_\lambda\, \mathfrak{h}'_\lambda}{N_\lambda},$$

or
$$e_{\lambda+1}\, \mathfrak{h}_{\lambda+1} = e'_\lambda\, \mathfrak{h}'_\lambda - t_\lambda \sin^2 \psi_\lambda, \quad \ldots \ldots \quad (25)$$

and
$$\mathfrak{h}^2_{\lambda+1} = \mathfrak{h}'^2_\lambda - 2e'_\lambda\, \mathfrak{h}'_\lambda\, t_\lambda + t^2_\lambda \sin^2 \psi_\lambda \quad . \quad . \quad (26)$$

are the equations of transference to the new surface.

To complete the calculations on the lines laid down in the previous Paper, formulæ are required for d_λ. Either of the formulæ given earlier,

$$d_\lambda \cos \psi_\lambda = t_\lambda - \tfrac{1}{2}\mathfrak{H}'_\lambda R_\lambda + \tfrac{1}{2}\mathfrak{H}^2_{\lambda+1} R_{\lambda+1}, \quad \ldots \ldots \quad (13)$$

or
$$d_\lambda = t_\lambda \cos \psi_\lambda - \sqrt{(\mathfrak{H}^2_\lambda - \mathfrak{h}'^2_\lambda)} + \sqrt{(\mathfrak{H}^2_{\lambda+1} - \mathfrak{h}^2_{\lambda+1})}, \quad (14)$$

may be used for this purpose. The latter is usually more convenient.

It should be noted that in the preliminary calculations to which the above equations relate, neither the sines of the angles involved nor the angles themselves need be found. The equations relate entirely to cosines with the exception of

$$\mu_\lambda \sin \phi'_\lambda = \mu_{\lambda-1} \sin \phi_\lambda,$$

and this equation is in practice squared and written as a cosine equation,

$$\cos^2 \phi'_\lambda = 1 - \frac{\mu^2_{\lambda-1}}{\mu^2_\lambda}(1 - \cos^2 \phi_\lambda).$$

The use of cosines instead of sines would be objectionable in a system in which reference to tables is involved, since a knowledge of the cosine of small angles does not determine the angle itself to high accuracy. This does not apply to the system outlined above, where the object is not to know the final angles of emergence, but to determine the K's, d's and the cosines of the various angles of incidence and refraction. The position of the emergent ray is obtained from a secondary calculation, involving these quantities, as shown in the earlier Paper.

As in the two dimensional case, the formulæ just obtained may be combined with those contained in the previous Paper. The equations of transference are easily found and need no discussion. It will be noted that many alternative forms can be given to the equations for performing the calculations con-

sistently with the conditions laid down earlier. The system adopted is to be regarded as an illustration of a possible arrangement, and is not necessarily the best of its class.

The operations may be summarised as follows: Given $\mathfrak{h}_\lambda^2$, $e_\lambda \mathfrak{h}_\lambda$, $\cos \psi_{\lambda-1}$, it is required to find $\mathfrak{h}^2_{\lambda+1}$, $e_{\lambda+1}\mathfrak{h}_{\lambda+1}$, $\cos\psi_\lambda$ and also d_λ, K_λ, $\cos\varphi_\lambda$, $\cos\varphi'_\lambda$.

$$\cos^2\varphi_\lambda=\cos^2\psi_{\lambda-1}-(\mathfrak{h}_\lambda^2 R_\lambda-2e_\lambda\mathfrak{h}_\lambda)R_\lambda \qquad \text{gives } \cos\varphi_\lambda \quad (21)$$

$$\mu_\lambda^2(1-\cos^2\varphi'_\lambda)=\mu^2_{\lambda-1}(1-\cos^2\varphi_\lambda) \qquad ,, \ \cos\varphi' \quad (10)$$

$$K_\lambda=(\mu_\lambda\cos\varphi'_\lambda-\varphi_{\lambda-1}\cos\varphi_\lambda)R_\lambda \qquad ,, \ K_\lambda \quad (8)$$

$$\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}_\lambda^2)}=\frac{\mathfrak{h}_\lambda^2 R_\lambda-2e_\lambda\mathfrak{h}_\lambda}{\cos\psi_{\lambda-1}+\cos\varphi_\lambda} \qquad ,, \ \mathfrak{H}_\lambda^2 \quad (19)$$

$$\mu_\lambda\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}'^2_\lambda)}=\mu_{\lambda-1}\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}_\lambda^2)}-\tfrac{1}{2}\mathfrak{H}_\lambda^2 K_\lambda \qquad ,, \ \mathfrak{h}'^2_\lambda \quad (23)$$

$$\cos\psi_\lambda=\cos\varphi'_\lambda+R_\lambda\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}'^2_\lambda)} \qquad ,, \ \cos\psi_\lambda \quad (22)$$

$$e'_\lambda\mathfrak{h}'_\lambda=\tfrac{1}{2}\mathfrak{H}_\lambda^2 R_\lambda-\cos\psi_\lambda\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}'^2_\lambda)} \qquad ,, \ e'_\lambda\mathfrak{h}'_\lambda \quad (24)$$

$$e_{\lambda+1}\mathfrak{h}_{\lambda+1}=e'_\lambda\mathfrak{h}'_\lambda-t_\lambda(1-\cos^2\psi_\lambda) \qquad ,, \ e_{\lambda+1}\mathfrak{h}_{\lambda+1} \quad (25)$$

$$\mathfrak{h}^2_{\lambda+1}=\mathfrak{h}'^2_\lambda-2e'_\lambda\mathfrak{h}'_\lambda t_\lambda+t_\lambda^2(1-\cos^2\psi_\lambda) \qquad ,, \ \mathfrak{h}^2_{\lambda+1} \quad (26)$$

$$d_\lambda=t_\lambda\cos\psi_\lambda-\sqrt{(\mathfrak{H}_\lambda^2-\mathfrak{h}'^2_\lambda)}+\sqrt{(\mathfrak{H}^2_{\lambda+1}-\mathfrak{h}^2_{\lambda+1})} \qquad ,, \ d_\lambda \quad (14)$$

It is evident on comparing these equations with those obtained in the earlier Paper that the conditions imposed have somewhat lengthened the calculations. The addition will, however, be found to be very slight. The appearance of the equations gives a misleading impression of complexity, but this is simply due to the notation employed. Two square roots have to be extracted to determine $\cos\varphi$ and $\cos\varphi'$, the same number as in the earlier system. All else is multiplication, division or addition. The trigonometrical equations of Von Seidel, which in appearance are remarkably simple, will be found to take up much more time. The latter involve at least 20 references to tables for each refracting surface, and the calculations when completed afford less information than can be derived from calculations on the algebraic system.

The earlier Paper already mentioned discussed the calculation of radial focal lines. It may be of interest to add the corresponding arrangement which has been found most convenient in dealing with the primary or transverse focal lines. As in the case of the radial focal line, the arrangement of the work resembles that followed for paraxial rays. Put

$$Y_\lambda=K_\lambda/\cos\varphi_\lambda\cos\varphi'_\lambda \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and let $\mathfrak{Y}_{1,\lambda}$, $\frac{\partial \mathfrak{Y}_{1,\lambda}}{\partial \mathfrak{Y}_\lambda}$ be defined by the equations,

$$\left.\begin{aligned} \mathfrak{Y}_{1\,\lambda} &= \frac{\cos\varphi_\lambda}{\cos\varphi_\lambda'}\mathfrak{Y}_{1,\lambda-1}+\mathfrak{Y}_\lambda\frac{\partial \mathfrak{Y}_{1,\lambda}}{\partial \mathfrak{Y}_\lambda} \\ \frac{\partial \mathfrak{Y}_{1,\lambda+1}}{\partial \mathfrak{Y}_{\lambda+1}} &= \frac{\cos\varphi_\lambda'}{\cos\varphi_\lambda}\ \frac{\partial \mathfrak{Y}_{1,\lambda}}{\partial \mathfrak{Y}_\lambda}-\frac{d_\lambda}{\mu_\lambda}K_{1\,\lambda} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (28)$$

where

$$\frac{\partial \mathfrak{Y}_{1,1}}{\partial \mathfrak{Y}_1}=1,\ \mathfrak{Y}_{1,1}=\mathfrak{Y}_1,\ \mathfrak{Y}_{1,0}=0.$$

The same value of $\mathfrak{Y}_{1,n}$ is obtained by starting the calculations at the other end of the system, using the formulæ,

$$\left.\begin{aligned} \mathfrak{Y}_{\lambda,n} &= \frac{\cos\varphi_\lambda'}{\cos\varphi_\lambda}\mathfrak{Y}_{\lambda+1,n}+\mathfrak{Y}_\lambda\frac{\partial \mathfrak{Y}_{\lambda\,n}}{\partial \mathfrak{Y}_\lambda} \\ \frac{\partial \mathfrak{Y}_{\lambda-1,n}}{\partial \mathfrak{Y}_{\lambda-1}} &= \frac{\cos\varphi_\lambda}{\cos\varphi_\lambda'}\ \frac{\partial \mathfrak{Y}_{\lambda,n}}{\partial \mathfrak{Y}_\lambda}-\frac{d_{\lambda-1}}{\mu_{\lambda-1}}\mathfrak{Y}_{\lambda,n} \end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad (29)$$

where $\quad \frac{\partial \mathfrak{Y}_{n,n}}{\partial \mathfrak{Y}_n}=1,\ \mathfrak{Y}_{n,n}=\mathfrak{Y}_n,\ \mathfrak{Y}_{n+1\,n}=0.$

The quantities defined by these equations have properties analogous to the corresponding paraxial functions. Rays arising from a point at a distance p_0 along the incident ray in the first medium measured in the direction in which the light travels from the point at which the first refraction takes place will be refracted through a point on the emergent ray at a distance, p_n, from the point of refraction in the last surface measured in the same direction, where

$$p_0=\mu_0\left(-\frac{\cos\varphi_1}{\cos\varphi_1'}\ \frac{\partial \mathfrak{Y}_{1,n}}{\partial \mathfrak{Y}_1}+\frac{1}{\mathfrak{D}}\right)/\mathfrak{Y}_{1,n} \quad . \quad . \quad . \quad . \quad (30)$$

and

$$p_n=\mu_n\left(\frac{\cos\varphi_n'}{\cos\varphi_n}\ \frac{\partial \mathfrak{Y}_{1\,n}}{\partial \mathfrak{Y}_n}-\mathfrak{D}\right)/\mathfrak{Y}_{1,n}, \quad . \quad . \quad . \quad . \quad (31)$$

provided the rays considered lie in an axial plane and lie close to the original ray throughout their passage through the optical system. Also if a small linear object lying in an axial plane and normal to the original ray meets the ray in the point distant p_0 from the first surface, its image formed by neighbouring rays in an axial plane will be normal to the emergent ray, will meet this ray in the point determined by p_n, and the linear magnification will be $\mathfrak{D}$.

Many other formulæ can be contrived for these calculations, and it is easy to put down generalised formulæ. It will be found that none of these are so convenient as those given above for general use, though in special cases they may offer distinct advantages. As an illustration, take the formulæ,

$$P_\lambda = K_\lambda$$

$$P_{,\lambda} = \cos^2 \varphi_\lambda P_{1,\lambda-1} + P_\lambda \frac{\partial P_{1,\lambda}}{\partial P_\lambda},$$

$$\frac{\partial P_{1\,\lambda+1}}{\partial P_{\lambda+1}} = \cos^2 \varphi_\lambda' \frac{\partial P_{1,\lambda}}{\partial P_\lambda} - \frac{d_\lambda}{\mu_\lambda} P_{1,\lambda},$$

with the symmetrical reverse formulæ,

$$P_{\lambda,n} = \cos^2 \varphi_\lambda' P_{\lambda+1,n} + P_\lambda \frac{\partial P_{\lambda\, n}}{\partial P_\lambda}$$

$$\frac{\partial P_{\lambda-1,n}}{\partial P_{\lambda-1}} = \cos^2 \varphi_\lambda \frac{\partial P_{\lambda,n}}{\partial P_\lambda} - \frac{d_{\lambda-1}}{\mu_{\lambda-1}} P_{\lambda,n}.$$

It may be shown that

$$P_{1,n} = \mathfrak{H}_{1,n} \cos \varphi_1 \cos \varphi_1' \cos \varphi_2 \cos \varphi_2' \ldots \cos \varphi_n \cos \varphi_n',$$

where the continued product on the right consists of the cosines of all the angles of incidence and emergence. Also

$$\cos^2 \varphi_n' \frac{\partial P_{1,n}}{\partial P_n} / P_{1,n} = \frac{\cos \varphi_n}{\cos \varphi_n} \cdot \frac{\partial \mathfrak{H}_{1,n}}{\partial \mathfrak{H}_n} / \mathfrak{H}_{1,n}.$$

Thus the terms involving the magnification and its reciprocal in the formulæ for conjugate distances would, with the P formulæ, require to be multiplied by the above continued product of cosines, and the labour of calculation in a complex system would be much increased.

It follows from the formulæ for conjugate distances that $\mathfrak{H}_{1,n}$ is the power of the system for a pencil of primary rays lying close to the given ray, just as $K_{1,n}$ is the power for a pencil of secondary rays. In the application of the formulæ for the primary rays it must be remembered that $\mathfrak{G}$ in general will differ from G for the same object and image points in a system corrected for astigmatism, because in the case of the former the dimensions of the object and image are not usually measured in planes normal to the rays. The appropriate value for $\mathfrak{G}$ is to be found by orthogonally projecting the actual dimensions of the object and image on to planes normal to the incident and emergent rays.

PUBLICATIONS OF THE PHYSICAL SOCIETY.

THE SCIENTIFIC PAPERS
OF THE LATE
SIR CHARLES WHEATSTONE, F.R.S.
Demy 8vo, cloth. Price 8s.; to Fellows, 4s.

Uniform with the above.
THE SCIENTIFIC PAPERS
OF
JAMES PRESCOTT JOULE, D.C.L., F.R.S.
Vol. I. 4 Plates and Portrait, price 12*s.*; to Fellows, 6*s.*
Vol. II. 3 Plates, price 8*s.*; to Fellows, 4*s.*

PHYSICAL MEMOIRS.
PART I.—VON HELMHOLTZ, On the Chemical Relations of Electrical Currents. Pp. 110. *Price* 4s.; *to Fellows*, 2*s.*

PART II.—HITTORF, On the Conduction of Electricity in Gases; PULUJ, On Radiant Electrode Matter. Pp. 222. *Price* 8*s.*; *to Fellows* 4*s.*

PART III.—VAN DER WAALS, On the Continuity of the Liquid and Gaseous States of Matter. Pp. 164. *Price* 8*s.*; *to Fellows*, 4*s.*

REPORT ON RADIATION AND THE QUANTUM-THEORY.
By J. H. JEANS, M.A., F.R.S.
Price 6*s.*; *to Fellows*, 3*s.* *Bound in Cloth*, 8*s.* 6*d.*; *to Fellows*, 5*s.* 6*d.*

PROCEEDINGS.
The "Proceedings" of the Physical Society can be obtained at the following prices:—

Vol. I. (3 parts) bound cloth, 15*s.*
Vols. II., IV., V., XXIII., XXV., XXVI., XXVII., XXVIII. & XXIX. (5 parts each), bound cloth, 23*s.*
Vols. III., VI. to XII. & XXII. (4 parts each), bound cloth, 19*s.*
Vol. XIII. (13 parts, each containing Abstracts), bound cloth (without Abstracts), 47*s.*
Vols. XIV. & XV. (12 parts, each containing Abstracts), bound cloth (without Abstracts), 23*s.*
Vols. XVI. & XIX. (8 parts each), bound cloth, 35*s.*
Vols. XVII., XVIII. & XXI. (7 parts each), bound cloth, 31*s.*
Vols. XX. & XXIV. (6 parts), bound cloth, 27*s.*

Most of the parts can be purchased separately, price 4*s.* by post 4*s.* 3*d.* Fellows can obtain the *Proceedings* (in parts) for their personal use at half the above prices.

ABSTRACTS OF PHYSICAL PAPERS
FROM FOREIGN SOURCES.
VOLS. I. (1895), II. (1896), and III. (1897), 15*s.* each; to Fellows, 7*s.* 6*d.* each.

Strong cloth cases for binding the "Proceedings," price 1s. 9d. each, post free

BLAKESLEY, T. H. A Table of Hyperbolic Sines and Cosines. *Price* 1*s.* 6*d.*; *to Fellows*, 9*d.*
LEHFELDT, R. A. A List of Chief Memoirs on the Physics of Matter. *Price* 2*s.*; *to Fellows*, 1*s.*

Applications for the above Publications should be sent direct to
FLEETWAY PRESS, LTD.,

CONTENTS.